

INDICES OF VECTOR FIELDS ON SINGULAR VARIETIES: AN OVERVIEW

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0 Introduction

The Poincaré-Hopf total index of a vector field with isolated singularities on a smooth, closed manifold M can be regarded as the obstruction to constructing a non-zero section of the tangent bundle TM . In this way it extends naturally to complex vector bundles in general and leads to the notion of Chern classes. When working with singular analytic varieties, it is thus natural to ask what should be the notion of “the index” of a vector field.

Indices of vector fields on singular varieties were first considered by M. H. Schwartz in [51, 52] in her study of Chern classes for singular varieties. For her purpose there was no point in considering vector fields in general, but only a special class of vector fields (and frames) that she called “radial”, which are obtained by the important process of *radial extension*. The generalisation of this index to other vector fields was defined independently in [38, 15, 56] (see also [2]), and its extension for frames in general was done in [6]. This index, that we call *Schwartz index*, is sometimes called “radial index” because it measures how far the vector field is from being radial. In [15, 2] this index is defined also for vector fields on real analytic varieties.

MacPherson in [46] introduced the local Euler obstruction, also for constructing Chern classes of singular complex algebraic varieties. In [8] this invariant was defined via vector fields, interpretation that was essential to prove (also in [8]) that the Schwartz classes of a singular variety coincide with MacPherson’s classes. This viewpoint brings the local Euler obstruction into the frame-work of “indices of vector fields on singular varieties” and yields to another index, that we may call *the local Euler obstruction* of the vector field at each isolated singularity; the Euler obstruction of the singular variety corresponding to the case of the radial vector field. This index relates to the previously mentioned Schwartz index by a formula known as the “Proportionality Theorem” of [8]. When the

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vector field is determined by the gradient of a function on the singular variety, this local Euler obstruction is the defect studied in [7].

On the other hand, one of the basic properties of the local index of Poincaré-Hopf is that it is stable under perturbations. In other words, if v is a vector field on an open set U in \mathbb{R}^n and $x \in U$ is an isolated singularity of v , and if we perturb v slightly, then its singularity at x may split into several singular points of the new vector field \hat{v} , but the sum of the indices of \hat{v} at these singular points equals the index of v at x . If we now consider an analytic variety V defined, say, by a holomorphic function $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ with an isolated critical point at 0, and if v is a vector field on V , non-singular away from 0, then one would like “the index” of v at 0 to be stable under small perturbations of both, the function f and the vector field v . The extension of this index to the case of vector fields on isolated complete intersection singularity germs (ICIS for short) is immediate. This leads naturally to another concept of index, now called the *GSV-index*, introduced in [53, 26, 56]. There is also the analogous index for continuous vector fields on real analytic varieties (see [2, 31, 32]).

One also has the virtual index, introduced in [42] for holomorphic vector fields; the extension to continuous vector fields is immediate and was done in [56, 6]. This index is defined via Chern-Weil theory. The idea is that the usual Poincaré-hopf index can be regarded as a localisation, at the singular points of a vector field, of the n^{th} -Chern class of a manifold. Similarly, for an ICIS $(V, 0)$ in \mathbb{C}^{n+k} , defined by functions $f = (f_1, \dots, f_k)$, one has a localisation at 0 of the top Chern class of the ambient space, defined by the gradient vector fields of the f_i and the given vector field, tangent to V . This localisation defines *the virtual index* of the vector field; this definition extends to a rather general setting, providing a topological way for looking at the top Chern class of the so-called virtual tangent bundle of singular varieties which are local complete intersections. In the case envisaged above, when $(V, 0)$ is an ICIS, this index coincides with the GSV-index.

Another remarkable property of the local index of Poincaré-Hopf is that in the case of germs of holomorphic vector fields in \mathbb{C}^n with an isolated singularity at 0, the local index equals the integer:

$$\dim \mathcal{O}_{\mathbb{C}^n, 0} / (X_1, \dots, X_n), \quad (*)$$

where (X_1, \dots, X_n) is the ideal generated by the components of the vector field. This and other facts motivated the search for algebraic formulae for the index of vector fields on singular varieties. The *homological index* of Gomez-Mont [25] is a beautiful answer to that search. It considers an isolated singularity germ $(V, 0)$ of any dimension, and a holomorphic vector field on V , singular only at 0. One has the Kähler differentials on V , and a Koszul complex $(\Omega_{V, 0}^\bullet, v)$:

$$0 \rightarrow \Omega_{V, 0}^n \rightarrow \Omega_{V, 0}^{n-1} \rightarrow \dots \rightarrow \mathcal{O}_{V, 0} \rightarrow 0,$$

where the arrows are given by contracting forms by the vector field v . The homological index of v is defined to be the Euler characteristic of this complex. When the ambient space V is smooth at 0, the complex is exact in all dimensions, except in degree 0 where the corresponding homology group has dimension equal to the local index of Poincaré-

Hopf of v . When $(V, 0)$ is a hypersurface germ, this index coincides with the GSV-index, but for more general singularities the homological index is still waiting to be understood!

In fact, in [21] there is given the corresponding notion of *homological index* for holomorphic 1-forms on singular varieties, and recent work of Schürmann throws light into this, yet mysterious, invariant.

When considering smooth (real) manifolds, the tangent and cotangent bundles are canonically isomorphic and it does not make much difference to consider either vector fields or 1-forms in order to define their indices and their relations with characteristic classes. When the ambient space is a complex manifold, this is no longer the case, but there are still ways for comparing indices of vector fields and 1-forms, and to use these to study Chern classes of manifolds. To some extent this is also true for singular varieties, but there are however important differences and each of the two settings has its own advantages.

The first time that indices of 1-forms on singular varieties appeared in the literature was in MacPherson's work [46], where he defined the local Euler obstruction in this way. But the systematic study of these indices was begun by W. Ebeling and S. Gusein-Zade in a series of articles (see for instance [16, 18, 19, 20]). This has been, to some extent, a study parallel to the one for vector fields, outlined in this article. Also along this lines is [10], which adapts to 1-forms the radial extension technique of M. H. Schwartz and proves the corresponding Proportionality Theorem.

Also, J. Schürmann in his book [49] introduces powerful methods to studying singular varieties via micro-local analysis and Lagrangian cycles, and much of the theory of indices of 1-forms can also be seen in that way. Furthermore, he has recently found a remarkable method for assigning an index of 1-forms to each constructible function on a Whitney stratified complex analytic space, in such a way that each of the known indices corresponds to a particular choice of a constructible function. This is closely related to MacPherson work in [46] for defining characteristic classes of singular varieties.

In this article we briefly review the various indices of vector fields on singular varieties. I am presently working with Jean-Paul Brasselet and Tatsuo Suwa writing [11], a monograph with a detailed account of all these indices, through the viewpoints of algebraic topology (obstruction theory) and differential geometry (Chern-Weil theory), together with their relations with Chern classes of singular varieties. This will include some applications of these indices to other fields of singularity theory.

This article grew from my talk in the singularities meeting at the CIRM in Luminy in celebration of the 60th anniversary of Jean-Paul Brasselet, and I want to thank the organizers for the invitations to participate in that meeting and to write these notes, particularly to Anne Pichon. I am also grateful to Tatsuo Suwa, Jean-Paul Brasselet and Jörg Schürmann for many helpful conversations.

1 The Schwartz index

Consider first the case when the ambient space is an affine irreducible complex analytic variety $V \subset \mathbb{C}^N$ of dimension $n > 1$ with an isolated singularity at 0. Let U be an open ball around $0 \in \mathbb{C}^N$, small enough so that every sphere in U centered at 0 meets V transversally (see [47]). For simplicity we restrict the discussion to U and set $V = V \cap U$. Let v_{rad} be a continuous vector field on $V \setminus \{0\}$ which is transversal (outwards-pointing) to all spheres around 0, and scale it so that it extends to a continuous section of $T\mathbb{C}^N|_V$ with an isolated zero at 0. We call v_{rad} a *radial vector field* at $0 \in V$. Notice v_{rad} can be further extended to a *radial* vector field $v_{rad}^\#$ on all of U , *i.e.* transversal to all spheres centered at 0. By definition *the Schwartz index* of v_{rad} is the Poincaré-Hopf index at 0 of the radial extension $v_{rad}^\#$, so it is 1. Of course we could have started with the zero-vector at 0, then extend this to v_{rad} on V as above, and then extend it further to all of U being transversal to all the spheres, getting the same answer; this is the viewpoint that generalises when the singular set of V has dimension more than 0.

Let us continue with the case when V has an isolated singularity at 0, and assume now that v is a continuous vector field on V with an isolated singularity at 0. By this we mean a continuous section v of $T\mathbb{C}^N|_V$ which is tangent to $V^* = V \setminus \{0\}$. We want to define *the Schwartz index* of v ; this index somehow measures the “radiality” of the vector field. It has various names in the literature (c.f. [38, 56, 17, 2]), one of them being *radial index*.

Let v_{rad} be a radial vector field at 0, *i.e.* v_{rad} is transversal, outwards-pointing, to the intersection of V with every sufficiently small sphere \mathbb{S}_ε centered at 0. We may now define the difference between v and v_{rad} at 0: consider small spheres $\mathbb{S}_\varepsilon, \mathbb{S}_{\varepsilon'}$; $\varepsilon > \varepsilon' > 0$, and let w be a vector field on the cylinder X in V bounded by the links $K_\varepsilon = \mathbb{S}_\varepsilon \cap V$ and $K_{\varepsilon'} = \mathbb{S}_{\varepsilon'} \cap V$, such that w has finitely many singularities in the interior of X , it restricts to v on K_ε and to v_{rad} on $K_{\varepsilon'}$. The *difference* $d(v, v_{rad}) = d(v, v_{rad}; V)$ of v and v_{rad} is:

$$d(v, v_{rad}) = \text{Ind}_{PH}(w; X),$$

the Poincaré-Hopf index of w on X . Then define *the Schwartz* (or *radial*) *index* of v at $0 \in V$ to be:

$$\text{Ind}_{Sch}(v, 0; V) = 1 + d(v, v_{rad}).$$

The following result is well known (see for instance [38, 17, 56, 2]). For vector fields with radial singularities, this is a special case of the work of M. H. Schwartz; the general case follows easily from this.

Theorem 1.1 *Let V be a compact complex analytic variety with isolated singularities q_1, \dots, q_r in a complex manifold M , and let v be a continuous vector field on V , singular at the q_i 's and possibly at some other isolated points in V . Let $\text{Ind}_{Sch}(v; V)$ be the sum of the Schwartz indices of v at the q_i plus its Poincaré-Hopf index at the singularities of v in the regular part of V . Then:*

$$\text{Ind}_{Sch}(v; V) = \chi(V).$$

The proof is fairly simple and we refer to the literature for details.

The idea for defining the Schwartz index in general, when the singular set has dimension more than 0, is similar in spirit to the case above, but it presents some technical difficulties. Consider a compact, complex analytic variety V of dimension n embedded in a complex manifold M , equipped with a Whitney stratification $\{V_\alpha\}_{\alpha \in A}$ adapted to V . The starting point to define the Schwartz index of a vector field is the *radial extension* introduced by M. H. Schwartz. To explain this briefly, let v be a vector field defined on a neighbourhood of 0 in the stratum V_α of V that contains 0. The fact that the stratification is Whitney implies (see [51, 52, 5] for details) that one can make a *parallel extension* of v to a stratified vector field v' on a neighbourhood of 0 in M . Now, if 0 is an isolated singularity of v on V_α , then v' will be singular in a disc of dimension $(\dim_{\mathbb{R}} M - \dim_{\mathbb{R}} V_\alpha)$, transversal to V_α in M at 0. So this extension is not good enough by itself. We must add to it another vector field v'' : the gradient of the square of the function “distance to” V_α , defined near 0. This vector field is transversal to the boundaries of all tubular neighbourhoods of V_α in M ; using the Whitney conditions we can make v'' be a continuous, stratified vector field near 0. The zeroes of v'' are the points in V_α . Adding v' and v'' at each point near 0 we get a stratified, continuous vector field $v^\#$ defined on a neighbourhood of 0 in M , which restricts to the given vector field v on V_α . This vector field has the additional property of being radial in all directions which are normal to the stratum V_α . In other words, if we take a small smooth disc Σ in M transversal to V_α at 0 of dimension complementary to that of V_α . Then the restriction of $v^\#$ to Σ can be projected into a vector field tangent to Σ with Poincaré-Hopf index 1 at 0. Hence the Poincaré-Hopf index of v on the stratum V_α equals the Poincaré-Hopf index of $v^\#$ in the ambient space M : this is a basic property of the vector fields obtained by radial extension.

Definition 1.2 The *Schwartz index* of v at $0 \in V_\alpha \subset V$ is defined to be the Poincaré-Hopf index at 0 of its radial extension $v^\#$ to a neighbourhood of 0 in M .

From the previous discussion we deduce:

Proposition 1.3 *If the stratum V_α has dimension > 0 , the Schwartz index of v equals the Poincaré-Hopf index of v at 0 regarded as a vector field on the stratum V_α .*

Now, more generally, let v be a stratified vector field on V with an isolated singularity at $0 \in V \subset M$. Let v_{rad} be a stratified radial vector field at 0, *i.e.* v_{rad} is transversal (outwards-pointing) to the intersection of V with every sufficiently small sphere \mathbb{S}_ε in M centered at 0, and it is tangent to each stratum. We define the difference between v and v_{rad} at 0 as follows. Consider sufficiently small spheres $\mathbb{S}_\varepsilon, \mathbb{S}_{\varepsilon'}$ in M , $\varepsilon > \varepsilon' > 0$, and put the vector field v on $K_\varepsilon = \mathbb{S}_\varepsilon \cap V$ and v_{rad} on $K_{\varepsilon'} = \mathbb{S}_{\varepsilon'} \cap V$. We now use the Schwartz’s technique of radial extension explained before, to get a stratified vector field w on the cylinder X in V bounded by the links K_ε and $K_{\varepsilon'}$, such that w extends v and v_{rad} , it has finitely many singularities in the interior of X and at each of these singular points its index in the stratum equals its index in the ambient space M (see [5] for details). The

difference of v and v_{rad} is defined as:

$$d(v, v_{rad}) = \sum \text{Ind}_{PH}(w; X),$$

where the sum on the right runs over the singular points of w in X and each singularity is being counted with the local Poincaré-Hopf index of w in the corresponding stratum. As in the work of M. H. Schwartz, we can check that this integer does not depend on the choice of w .

Definition 1.4 *The Schwartz (or radial) index of v at $0 \in V$ is:*

$$\text{Ind}_{Sch}(v, 0; V) = 1 + d(v, v_{rad}).$$

It is clear that if V is smooth at 0 then this index coincides with the usual Poincaré-Hopf index; it also coincides with the index defined above when 0 is an isolated singularity of V and with the usual index of M. H. Schwartz for vector fields obtained by radial extension. In order to give a unified picture of what this index measures in the various cases, it is useful to introduce a concept that picks up one of the essential properties of the vector fields obtained by radial extension:

Definition 1.5 A stratified vector field on V is *normally radial* at $0 \in V_\alpha$ if it is radial in the direction of each stratum $V_\beta \neq V_\alpha$ containing 0 in its closure.

In other words, v is normally radial if its projection to each small disc Σ around 0, which is transversal to V_α at 0 and has dimension $(\dim_{\mathbb{R}} M - \dim_{\mathbb{R}} V_\alpha)$, is a radial vector field in Σ , *i.e.* it is transversal to each sphere in Σ centered at 0. The vector fields obtained by radial extension satisfy this condition at all points.

The proof of the following proposition is immediate from the definitions.

Proposition 1.6 *Let v be a stratified vector field on V with an isolated singularity at 0, and let V_α be the Whitney stratum that contains 0. If v is normally radial at 0, then its Schwartz index $\text{Ind}_{Sch}(v, 0; V)$ equals its Poincaré-Hopf index $\text{Ind}_{PH}(v, 0; V_\alpha)$ in V_α . Otherwise, its Schwartz index $\text{Ind}_{Sch}(v, 0; V)$ is the sum:*

$$\text{Ind}_{Sch}(v, 0; V) = \text{Ind}_{PH}(v, 0; V_\alpha) + \sum_{\beta \neq \alpha} d(v, v_{rad}; V_\beta),$$

where $\text{Ind}_{PH}(v, 0; V_\alpha)$ is defined to be 1 if the stratum V_α has dimension 0, and the sum in the right runs over all strata that contain V_α in their closures; $d(v, v_{rad}; V_\beta)$ is the difference in each stratum V_β between v and a stratified radial vector field v_{rad} at 0.

2 The local Euler obstruction

Let $(V, 0)$ be a reduced, pure-dimensional complex analytic singularity germ of dimension n in an open set $U \subset \mathbb{C}^N$. Let $G(n, N)$ denote the Grassmanian of complex n -planes in \mathbb{C}^N . On the regular part V_{reg} of V there is a map $\sigma : V_{reg} \rightarrow U \times G(n, N)$ defined by $\sigma(x) = (x, T_x(V_{reg}))$. The *Nash transformation* (or *Nash blow up*) \tilde{V} of V is the closure of $\text{Im}(\sigma)$ in $U \times G(n, N)$. It is a (usually singular) complex analytic space endowed with an analytic projection map

$$\nu : \tilde{V} \rightarrow V$$

which is a biholomorphism away from $\nu^{-1}(\text{Sing}(V))$, where $\text{Sing}(V) := V - V_{reg}$. Notice each point $y \in \text{Sing}(V)$ is being replaced by all limits of planes $T_{x_i} V_{reg}$ for sequences $\{x_i\}$ in V_{reg} converging to x .

Let us denote by $U(n, N)$ the tautological bundle over $G(n, N)$ and denote by \mathbb{U} the corresponding trivial extension bundle over $U \times G(n, N)$. We denote by π the projection map of this bundle. Let \tilde{T} be the restriction of \mathbb{U} to \tilde{V} , with projection map π . The bundle \tilde{T} on \tilde{V} is called *the Nash bundle* of V . An element of \tilde{T} is written (x, T, v) where $x \in U$, T is a d -plane in \mathbb{C}^N and v is a vector in T . We have maps:

$$\tilde{T} \xrightarrow{\pi} \tilde{V} \xrightarrow{\nu} V,$$

where π is the projection map of the Nash bundle over the Nash blow up \tilde{V} .

Let us consider a complex analytic stratification $(V_\alpha)_{\alpha \in A}$ of V satisfying the Whitney conditions. Adding the stratum $U \setminus V$ we obtain a Whitney stratification of U . Let us denote by $TU|_V$ the restriction to V of the tangent bundle of U . We know that a stratified vector field v on V means a continuous section of $TU|_V$ such that if $x \in V_\alpha \cap V$ then $v(x) \in T_x(V_\alpha)$. The Whitney condition (a) implies that given $x \in \text{Sing}(V)$, any limit \mathcal{T} of tangent spaces of points in $V_{reg} = V - \text{Sing}(V)$ converging to x contains the tangent space $T_x V_\alpha$ where V_α is the stratum that contains x . Hence one has the following lemma of [8]:

Lemma 2.1 *Every stratified vector field v on a set $A \subset V$ has a canonical lifting to a section \tilde{v} of the Nash bundle \tilde{T} over $\nu^{-1}(A) \subset \tilde{V}$.*

Now consider a stratified radial vector field $v(x)$ in a neighborhood of $\{0\}$ in V ; *i.e.* there is ε_0 such that for every $0 < \varepsilon \leq \varepsilon_0$, $v(x)$ is pointing outwards the ball \mathbb{B}_ε over the boundary $V \cap \mathbb{S}_\varepsilon$ with $\mathbb{S}_\varepsilon := \partial \mathbb{B}_\varepsilon$. Recall that, essentially by the Theorem of Bertini-Sard (see [47]), for ε small enough the spheres \mathbb{S}_ε are transverse to the strata $(V_\alpha)_{\alpha \in A}$.

One has the following interpretation of the local Euler obstruction [8]. We refer to [46] for the original definition which uses 1-forms instead of vector fields.

Definition 2.2 Let v be a stratified radial vector field on $V \cap \mathbb{S}_\varepsilon$ and \tilde{v} the lifting of v on $\nu^{-1}(V \cap \mathbb{S}_\varepsilon)$ to a section of the Nash bundle. The *local Euler obstruction* (or simply the Euler obstruction) $\text{Eu}_V(0)$ is defined to be the obstruction to extending \tilde{v} as a nowhere zero section of \tilde{T} over $\nu^{-1}(V \cap \mathbb{B}_\varepsilon)$.

More precisely, let $\mathcal{O}(\tilde{v}) \in H^{2d}(\mathbb{B}_\varepsilon, \nu^{-1}(V \cap \mathbb{S}_\varepsilon))$ be the obstruction cocycle for extending \tilde{v} as a nowhere zero section of \tilde{T} inside $\nu^{-1}(V \cap \mathbb{B}_\varepsilon)$, where \mathbb{B}_ε is a small ball around 0 and \mathbb{S}_ε is its boundary. The local Euler obstruction $Eu_V(0)$ is the evaluation of $\mathcal{O}(\tilde{v})$ on the fundamental class of the pair $(\nu^{-1}(V \cap \mathbb{B}_\varepsilon), \nu^{-1}(V \cap \mathbb{S}_\varepsilon))$. The Euler obstruction is an integer.

The following result summarises some basic properties of the Euler obstruction:

Theorem 2.3 *The Euler obstruction satisfies:*

- i. $Eu_V(0) = 1$ if 0 is a regular point of V ;
- ii. $Eu_{V \times V'}(0 \times 0') = Eu_V(0) \cdot Eu_{V'}(0')$;
- iii. If V is locally reducible at 0 and V_i are its irreducible components, then $Eu_V(0) = \sum Eu_{V_i}(0)$;
- iv. $Eu_V(0)$ is a constructible function on V , in fact it is constant on Whitney strata.

These statements are all contained in [46], except for the second part of (iv) which is not explicitly stated there and we refer to [8, 44] for a detailed proof.

More generally, for every point $x \in V$, we will denote by $V_\alpha(x)$ the stratum containing x . Now suppose v is a stratified vector field on a small disc \mathbb{B}_x around $x \in V$, and v has an isolated singularity at x . By 2.1 we have that v can be lifted to a section \tilde{v} of the Nash bundle \tilde{T} of V over $\nu^{-1}(\mathbb{B}_x \cap V)$ and \tilde{v} is never-zero on $\nu^{-1}(\partial\mathbb{B}_x \cap V)$. The obstruction for extending \tilde{v} without singularity to the interior of $\nu^{-1}(\mathbb{B}_x \cap V)$ is a cohomology class in $H^{2n}(\nu^{-1}(\mathbb{B}_x \cap V), \nu^{-1}(\partial\mathbb{B}_x \cap V))$; evaluating this class in the fundamental cycle $[\mathbb{B}_x, \partial\mathbb{B}_x]$ one gets an index $Eu(v, x; V) \in \mathbb{Z}$ of v at x . If v is radial at x then $Eu(v, x; V)$ is by definition the local Euler obstruction of V at x , $Eu_V(x)$.

Definition 2.4 The integer $Eu(v, x; V)$ is the (local) Euler obstruction of the stratified vector field v at $x \in V$.

As mentioned in the introduction, this index is related to the Schwartz index by the Proportionality Theorem of [8]. To state this result, recall that we introduced in section 1 the concept of normally radial vector fields, which essentially characterises the vector fields obtained by radial extension.

Theorem 2.5 (Proportionality Theorem [8]) *Let v be a stratified vector field on V which is normally radial at a singularity $0 \in V_\alpha$. Then one has:*

$$Eu(v, 0; V) = \text{Ind}_{Sch}(v, 0) \cdot Eu_V(0)$$

where $Eu_V(0)$ is the Euler obstruction of V at 0 and $\text{Ind}_{Sch}(v, 0)$ is the Schwartz index of v at 0.

In short, this theorem says that the obstruction $Eu(v, 0; V)$ to extend the lifting \tilde{v} as a section of the Nash bundle inside $\nu^{-1}(V \cap \mathbb{B}_\varepsilon(p))$ is proportional to the Schwartz index of v at 0, the proportionality factor being precisely the local Euler obstruction. We refer to [12] for a short proof of this theorem.

The invariant $Eu(v, 0; V)$ was studied in [7] when v is the “gradient vector field” ∇_f of a function f on V . More precisely, if V has an isolated singularity at 0 and the (real or complex valued) differentiable function f has an isolated critical point at 0, then v is truly the (complex conjugate if f is complex valued) gradient of the restriction of f to $V \setminus \{0\}$. In general, if V has a non-isolated singularity at 0 but f has an isolated critical point at 0 (in the stratified sense [33, 43]), then v is obtained essentially by projecting the gradient vector field of f to the tangent space of the strata in V , and then using the Whitney conditions to put these together in a continuous, stratified vector field. One may define this invariant even if f has non-isolated critical points, using intersections of characteristic cycles (see [7]), and it is a measure of how far the germ $(V, 0)$ is from satisfying the local Euler condition (in bivariant theory) with respect to the function f . Thus it was called in [7] *the Euler defect* of f at $(V, 0)$. The Euler obstruction of MacPherson corresponds to the case when f is the function distance to 0. As noticed in [20], this invariant can be also defined using the 1-form df instead of the gradient vector field. This avoids several technical difficulties and is closer to MacPherson’s original definition of the local Euler obstruction.

In [54] it is proved that if f has an isolated critical point at $0 \in V$ (in the stratified sense), then its “defect” equals the number of critical points in the regular part of V of a morsification of f . This fact can also be deduced easily from [48].

3 The GSV-index

Let us denote by $(V, 0)$ the germ of a complex analytic n -dimensional, isolated complete intersection singularity, defined by a function

$$f = (f_1, \dots, f_k) : (\mathbb{C}^{n+k}, 0) \rightarrow (\mathbb{C}^k, 0),$$

and let v be a continuous vector field on V singular only at 0. If $n = 1$ we further assume (for the moment) that V is irreducible. We use the notation of [45]: an ICIS means an isolated complete intersection singularity.

Since 0 is an isolated singularity of V , it follows that the (complex conjugate) gradient vector fields $\{\overline{\nabla}f_1, \dots, \overline{\nabla}f_k\}$ are linearly independent everywhere away from 0 and they are normal to V . Hence the set $\{v, \overline{\nabla}f_1, \dots, \overline{\nabla}f_k\}$ is a $(k+1)$ -frame on $V \setminus \{0\}$. Let $K = V \cap \mathbb{S}_\varepsilon$ be the link of 0 in V . It is an oriented, real manifold of dimension $(2n-1)$ and the above frame defines a map

$$\phi_v = (v, \overline{\nabla}f_1, \dots, \overline{\nabla}f_k) : K \rightarrow W_{k+1}(n+k),$$

into the Stiefel manifold of complex $(k+1)$ -frames in \mathbb{C}^{n+k} . Since $W_{k+1}(n+k)$ is simply connected, its first non-zero homology group is in dimension $(2n-1)$ and it is isomorphic

to \mathbb{Z} . Hence the map ϕ_v has a well defined degree $\deg(\phi_v) \in \mathbb{Z}$. To define it we notice that $W_{k+1}(n+k)$ is a fibre bundle over $W_k(n+k)$ with fibre the sphere \mathbb{S}^{2n-1} ; if (e_1, \dots, e_{n+k}) is the canonical basis of \mathbb{C}^{n+k} , then the fiber γ over the k -frame (e_1, \dots, e_k) determines the canonical generator $[\gamma]$ of $H_{(2n-1)}(W_k(n+k)) \cong \mathbb{Z}$. If $[K]$ is the fundamental class of K , then $(\phi_v)_*[K] = \lambda \cdot [\gamma]$ for some integer λ . Then the degree of ϕ_v is defined by:

$$\deg(\phi_v) = \lambda.$$

Alternatively one can prove that every map from a closed oriented $(2n-1)$ -manifold into $W_{k+1}(n+k)$ factors by a map into the fibre $\gamma \cong \mathbb{S}^{2n-1}$, essentially by transversality. Hence ϕ_v represents an element in $\pi_{2n-1}W_{k+1}(n+k) \cong \mathbb{Z}$, so ϕ_v is classified by its degree.

Definition 3.1 The GSV-index of v at $0 \in V$, $\text{Ind}_{\text{GSV}}(v, 0; V)$, is the degree of the above map ϕ_v .

This index depends not only on the topology of V near 0, but also on the way V is embedded in the ambient space. For instance the singularities in \mathbb{C}^3 defined by

$$\{x^2 + y^7 + z^{14} = 0\} \text{ and } \{x^3 + y^4 + z^{12} = 0\},$$

are orientation preserving homeomorphic, but one can prove that the GSV-index of the radial vector field is 79 in the first case and 67 in the latter; this follows from the fact (see 3.2 below) that for radial vector fields the GSV-index is $1 + (-1)^{\dim V} \mu$, where μ is the Milnor number, which in the examples above is known to be 78 and 66 respectively.

We recall that one has a Milnor fibration associated to the function f , see [47, 35, 45] and the Milnor fibre F can be regarded as a compact $2n$ -manifold with boundary $\partial F = K$. Moreover, by the Transversal Isotopy Lemma (see for instance [1]) there is an ambient isotopy of the sphere \mathbb{S}_ε taking K into ∂F , which can be extended to a collar of K , which goes into a collar of ∂F in F . Hence v can be regarded as a non-singular vector field on ∂F .

Theorem 3.2 *This index has the following properties:*

(i) *The GSV-index of v at 0 equals the Poincaré-Hopf index of v in the Milnor fibre:*

$$\text{Ind}_{\text{GSV}}(v, 0; V) = \text{Ind}_{\text{PH}}(v, F).$$

(ii) *If v is everywhere transversal to K , then*

$$\text{Ind}_{\text{GSV}}(v, 0; V) = 1 + (-1)^n \mu.$$

(iii) *One has:*

$$\text{Ind}_{\text{GSV}}(v, 0; V) = \text{Ind}_{\text{Sch}}(v, 0; V) + (-1)^n \mu,$$

where μ is the Milnor number of 0 and Ind_{Sch} is the Schwartz index.

Notice that the last statement says that the Milnor number of f equals (up to sign) the difference of the Schwartz and GSV indices of every vector field on V with an isolated singularity (cf. [21]).

In [9] there is a generalisation of this index to the case when the variety V has non-isolated singularities, but the vector field is stratified and it has an isolated singularity. In [2] is studied the real analytic setting and relations with other invariants of real singularities are given.

If V has dimension 1 and is not irreducible, then the GSV-index of vector fields on V was actually introduced by M. Brunella in [13, 14] and by Khanedani-Suwa [36], in relation with the geometry of holomorphic 1-dimensional foliations on complex surfaces. In this case one has two possible definitions of the index: as the Poincaré-Hopf index of an extension of the vector field to a Milnor fibre, or as the sum of the degrees in 3.1 corresponding to the various branches of V . One can prove [59, 2] that for plane curves these integers differ by the intersection numbers of the branches of V .

4 The Virtual Index

We now let V be a compact local complete intersection of dimension n in a manifold M of dimension $m = n + k$, defined as the zero set of a holomorphic section s of a holomorphic vector bundle E of rank k over M . The singular set of V , $Sing(V)$, may have dimension ≥ 0 . Let v be a C^∞ vector field on V . We denote by Σ the singular set of v , which is assumed to consist of $Sing(V)$ and possibly some other connected components in the regular part of V , disjoint from $Sing(V)$.

The virtual index is an invariant that assigns an integer to each connected component S of Σ . When S consists of one point, this index coincides with the GSV index, and for a component $S \subset V_{reg}$ this is just the sum of the local indices of the singularities into which S splits under a morsification of v .

Given a connected component S of Σ , the idea to define the virtual index $\text{Ind}_{Vir}(v; S)$ of v at S is to localize at S a certain characteristic class. We know that if V is smooth, then the usual Poincaré-Hopf local index can be regarded as the top Chern class $c_n(TD, v)$ of the tangent bundle of a small disc around the singular point of the vector field, relative to the section v given by the vector field on the boundary of D . In other words, the Poincaré-Hopf local index is obtained by localizing at 0 the top Chern class $c_n(TD)$ using the vector field v . We can of course replace the point 0 by a component S of Σ contained in V_{reg} ; in this case we replace D by a compact tubular neighbourhood \mathcal{T} of S in V and we localize $c_n(TV)|_{\mathcal{T}}$ at S using the vector field v , which is assumed to be non-singular on $\mathcal{T} \setminus S$. This means that we consider the Chern class $c_n(TV|_{\mathcal{T}})$ relative to v on $\mathcal{T} \setminus S$. The class we get lives in $H^{2n}(\mathcal{T}, \mathcal{T} \setminus S) \cong H_0(S) \cong \mathbb{Z}$. The integer that we get in this way is the Poincaré-Hopf index of v at S , $\text{Ind}_{PH}(v; S)$, *i.e.* the number of singularities of a generic perturbation of v inside \mathcal{T} , counted with signs.

The question now is what to do when S is contained in the singular set of V , so there is not a tangent bundle. The idea to define the virtual index is to make a similar

“localisation” using the vector field and the *virtual tangent bundle* of V , defined below.

To define this bundle we notice that the restriction $E|_{V_{reg}}$ coincides with the (holomorphic) normal bundle $N(V_{reg})$ of the regular part $V_{reg} = V - Sing(V)$. We denote by TM the holomorphic tangent bundle of M and we set $N = E|_V$.

Definition 4.1 (c.f. [24]) The *virtual tangent bundle* of V is

$$\tau(V) = TM|_V - N,$$

regarded as an element in the complex K-theory $KU(V)$.

It is known that the equivalence class of this virtual bundle does not depend on the choice of the embedding of V in M .

We denote by

$$c_*(TM|_V) = 1 + c_1(TM|_V) \dots + c_m(TM|_V),$$

and

$$c_*(N) = 1 + c_1(N) \dots + c_k(N),$$

the total Chern classes of these bundles. These are elements in the cohomology ring of V and can be inverted, *i.e.* there is a unique class $c_*(N)^{-1} \in H^*(V)$ such that

$$c_*(N) \cdot c_*(N)^{-1} = c_*(N)^{-1} \cdot c_*(N) = 1.$$

Using this one has the total Chern class of the virtual tangent bundle defined in the usual way:

$$c_*(\tau(V)) = c_*(TM|_V) \cdot c_*(N)^{-1} \in H^*(V).$$

The i^{th} Chern class of $TM|_V - N$ is by definition the component of $c_*(\tau(V))$ in dimension $2i$, for $i = 1, \dots, n$.

It is clear that if V is smooth, then its virtual tangent bundle is equivalent in $KU(V)$ to its usual tangent bundle, and the Chern classes of the virtual tangent bundle are the usual Chern classes.

Consider the component $c_n(\tau(V))$ of $c_*(\tau(V))$ in dimension $2n$. This is the top Chern class of the virtual tangent bundle. As we said before, the idea to define the local index of the vector field v at a component S of $Sing(V)$ is to localize $c_n(\tau(V))$ at S using v . For this one needs to explain how to localize the Chern classes of the virtual tangent bundle. This is carefully done in [59], and we refer to that text for a detailed account on the subject, particularly in relation with indices of vector fields.

In the particular case when the component S has dimension 0, so that we can assume we have a local ICIS germ $(V, 0)$ of dimension n in \mathbb{C}^{n+k} , defined by functions

$$f = (f_1, \dots, f_k) : U \subset \mathbb{C}^{n+k} \rightarrow \mathbb{C}^k,$$

with U an open set in \mathbb{C}^{n+k} , one has that the virtual tangent bundle of V is:

$$\tau(V) = T\mathbb{C}^{n+k}|_V - (V \times T\mathbb{C}^k).$$

If \mathbb{B} denotes a small ball in U around 0, then one has the Chern class $c_{n+k}(T\mathbb{B}|_V)$ relative to the $(k+1)$ -frame $(v, \bar{\nabla}f_1, \dots, \bar{\nabla}f_k)$ on $\partial\mathbb{B} \cap V$. This is a cohomology class in $H^{2n+2k}(\mathbb{B} \cap V, \partial\mathbb{B} \cap V)$, and one can prove that its image in $\cong H_0(\mathbb{B}) \cong \mathbb{Z}$ under the Alexander homomorphism is the virtual index of v at 0 (see [42, 56, 59]); which in this case coincides with the GSV-index.

5 The Homological Index

The basic references for this section are the articles by Gomez-Mont and various co-authors, see [25] and also [3, 27, 28, 29, 30]. There are also important algebraic formulas for the index of holomorphic vector fields (and 1-forms) given by various authors, as for instance in [42] (see also [26, 39, 40, 41]). In the real analytic case, interesting algebraic formulas for the index are given in [15, 31, 32], which generalize to singular hypersurfaces the remarkable formula of Eisenbud-Levine and Khimshiashvili [22, 37], that expresses the index of an analytic vector field in \mathbb{R}^m as the signature of an appropriate bilinear form. Here we only describe (briefly) the homological index of holomorphic vector fields.

Let $(V, 0) \subset (\mathbb{C}^N, 0)$ be the germ of a complex analytic (reduced) variety of pure dimension n with an isolated singular point at the origin. A vector field v on $(V, 0)$ can always be defined as the restriction to V of a vector field \tilde{v} in the ambient space which is tangent to $V \setminus \{0\}$; v is holomorphic if \tilde{v} can be chosen to be holomorphic. So we may write v as $v = (v_1, \dots, v_N)$ where the v_i are restriction to V of holomorphic functions on a neighbourhood of 0 in $(\mathbb{C}^N, 0)$.

It is worth noting that given every space V as above, there are always holomorphic vector fields on V with an isolated singularity at 0. This (non-trivial) fact is indeed a weak form of stating a stronger result ([3, 2.1, p. 19]): in the space $\Theta(V, 0)$ of germs of holomorphic vector fields on V at 0, those having an isolated singularity form a connected, dense open subset $\Theta_0(V, 0)$. Essentially the same result implies also that every $v \in \Theta_0(V, 0)$ can be extended to a germ of holomorphic vector field in \mathbb{C}^N with an isolated singularity, though it can possibly be also extended with a singular locus of dimension more than 0, a fact that may be useful for explicit computations (c.f. [25]).

A (germ of) holomorphic j -form on V at 0 means the restriction to V of a holomorphic j -form on a neighbourhood of 0 in \mathbb{C}^N ; two such forms in \mathbb{C}^N are equivalent if their restrictions to V coincide on a neighbourhood of $0 \in V$. We denote by $\Omega_{V,0}^j$ the space of all such forms (germs); these are the Kähler differential forms on V at 0. So, $\Omega_{V,0}^0$ is the local structure ring $\mathcal{O}_{(V,0)}$ of holomorphic functions on V at 0, and each $\Omega_{V,0}^j$ is a $\Omega_{V,0}^0$ module. Notice that if the germ of V at 0 is determined by (f_1, \dots, f_k) then one has:

$$\Omega_{V,0}^j := \frac{\Omega_{\mathbb{C}^N,0}^j}{(f_1 \Omega_{\mathbb{C}^N,0}^j + df_1 \wedge \Omega_{\mathbb{C}^N,0}^{j-1}, \dots, f_k \Omega_{\mathbb{C}^N,0}^j + df_k \wedge \Omega_{\mathbb{C}^N,0}^{j-1})},$$

where d is the exterior derivative.

Now, given a holomorphic vector field \tilde{v} at $0 \in \mathbb{C}^N$ with an isolated singularity at the origin, and a Kähler form $\omega \in \Omega_{\mathbb{C}^N,0}^j$, we can always contract ω by v in the usual way,

thus getting a Kähler form $i_v(\omega) \in \Omega_{\mathbb{C}^n,0}^{j-1}$. If $v = \tilde{v}|_V$ is tangent to V , then contraction is well defined at the level of Kähler forms on V at 0 and one gets a complex $(\Omega_{V,0}^\bullet, v)$:

$$0 \rightarrow \Omega_{V,0}^n \rightarrow \Omega_{V,0}^{n-1} \rightarrow \dots \rightarrow \mathcal{O}_{V,0} \rightarrow 0,$$

where the arrows are contraction by v and n is the dimension of V ; of course one also has Kähler forms of degree $> n$, but those forms do not play a significant role here. We consider the homology groups of this complex:

$$H_j(\Omega_{V,0}^\bullet, v) = \frac{\ker(\Omega_{V,0}^j \rightarrow \Omega_{V,0}^{j-1})}{\operatorname{Im}(\Omega_{V,0}^{j+1} \rightarrow \Omega_{V,0}^j)}$$

An important observation in [25] is that if V is regular at 0, so that its germ at 0 is that of \mathbb{C}^n at the origin, and if $v = (v_1, \dots, v_n)$ has an isolated singularity at 0, then this is the usual Koszul complex (see for instance [34, p. 688]), so that all its homology groups vanish for $j > 0$, while

$$H_0(\Omega_{V,0}^\bullet, v) \cong \mathcal{O}_{\mathbb{C}^n,0} / (v_1, \dots, v_n).$$

In particular the complex is exact when $v(0) \neq 0$. Since the contraction maps are $\mathcal{O}_{V,0}$ -modules maps, this implies that if V has an isolated singularity at the origin, then the homology groups of this complex are concentrated at 0, and they are finite dimensional because the sheaves of Kähler forms on V are coherent. Hence, for V a complex analytic affine germ with an isolated singularity at 0 and v a holomorphic vector field on V with an isolated singularity at 0, it makes sense to define:

Definition 5.1 The *homological index* $\operatorname{Ind}_{\operatorname{hom}}(v, 0; V)$ of the holomorphic vector field v on $(V, 0)$ is the Euler characteristic of the above complex:

$$\operatorname{Ind}_{\operatorname{hom}}(v, 0; V) = \sum_{i=0}^n (-1)^i h_i(\Omega_{V,0}^\bullet, v),$$

where $h_i(\Omega_{V,0}^\bullet, v)$ is the dimension of the corresponding homology group as a vector space over \mathbb{C} .

We recall that an important property of the Poincaré-Hopf local index is its stability under perturbations. This means that if we perturb v slightly in a neighbourhood of an isolated singularity, then this zero of v may split into a number of isolated singularities of the new vector field v' , whose total number (counted with their local indices) is the index of v . When the ambient space V has an isolated singularity at 0, then every vector field on V necessarily vanishes at 0, since in the ambient space the vector field defines a local 1-parameter family of diffeomorphisms. Hence every perturbation of v producing a vector field tangent to V must also vanish at 0, but new singularities may arise with this perturbation. The homological index also satisfies the stability under such perturbations. This is called the “Law of Conservation of Number” in [25, 28]:

Theorem 5.2 (Gomez-Mont [25, Theorem 1.2]) *For every holomorphic vector field v' on V sufficiently close to v one has:*

$$\mathrm{Ind}_{\mathrm{hom}}(v, 0; V) = \mathrm{Ind}_{\mathrm{hom}}(v', 0; V) + \sum \mathrm{Ind}_{PH}(v') ,$$

where Ind_{PH} is the local Poincaré-Hopf index and the sum on the right runs over the singularities of v' at regular points of V near 0.

This result is a special case of a more general theorem in [28].

This theorem is a key property of the homological index. In particular this allows us to identify this index with the GSV-index when $(V, 0)$ is a hypersurface germ [25]. In fact, it is easy to see that the GSV-index also satisfies the above “Law of Conservation of Number” for vector fields on complete intersection germs. This implies that if both indices coincide for a given vector field on $(V, 0)$, then they coincide for every vector field on $(V, 0)$, since the space $\Theta_0(V, 0)$ is connected. Hence, in order to prove that both indices coincide for all vector fields on hypersurface (or complete intersection) germs, it is enough to show that given every such germ, there exists a holomorphic vector field v for which the GSV and homological indices coincide. This is what Gomez-Mont does in [25]. For that, he first gives a very nice algebraic formula to compute the homological index of vector fields on hypersurface singularities, which he then uses to perform explicit computations and prove that, for holomorphic vector fields on hypersurface singularities, the homological index coincides with the GSV index. It is not known whether or not these indices coincide on complete intersection germs in general (c.f. [21]).

6 Relations with Chern classes of singular varieties

The local index of Poincaré-Hopf is the most basic invariant of a vector field at an isolated singularity, and the theorem of Poincaré-Hopf about the total index of a vector field on a manifold is a fundamental result, giving rise, in particular, to obstruction theory and the theory of characteristic classes, such as the Chern classes of complex manifolds.

In the case of singular varieties, there are several definitions of characteristic classes, given by various authors. Somehow they correspond to the various extensions one has of the concept of “tangent bundle” as we go from manifolds to singular varieties, and they are closely related to the indices of vector fields discussed above.

The first one is due to M.H. Schwartz in [51, 52], considering a singular complex analytic variety V embedded in a smooth complex manifold M which is equipped with a Whitney stratification adapted to V ; she then replaces the tangent bundle by the union of tangent bundles of all the strata in V , and considers a class of stratified frames to define characteristic classes of V , which do not depend on M nor on the various choices. These classes live in the cohomology groups of M with support in V , *i.e.* $H^*(M, M \setminus V; \mathbb{Z})$, and they are equivalent to the usual Chern classes when V is non-singular. The top degree Schwartz class is defined precisely using the Schwartz index presented in Section 1: consider a Whitney stratification of M adapted to V , a triangulation (K) compatible

with the stratification, and the dual cell decomposition (D) (c.f. [4, 5] for details). By construction, the cells of (D) are transverse to the strata. Consider now a stratified vector field v on V obtained by radial extension. Then (see [51, 52, 8, 5]) the radial extension technique allows us to construct a vector field on a regular neighborhood U of V in M , union of (D) -cells, which is normally radial (in the sense of section 1), and has at most a singular point at the barycenter of each (D) -cell of top dimension $2m$, m being the complex dimension of M . This defines a cochain in the usual way, by assigning to each such cell the Schwartz index of this vector field (*i.e.* its Poincaré-Hopf index in the ambient space). This cochain is a cocycle that represents a cohomology class in $H^*(U, U \setminus V) \cong H^*(M, M \setminus V)$, and this class is by definition the top Schwartz class of V . Its image in $H_0(V)$ under Alexander duality gives the Euler-Poincaré characteristic of V . The Schwartz classes of lower degrees are defined similarly, considering stratified r -frames, $r = 1, \dots, n = \dim V$, defined by radial extension on the $(2m - 2r + 2)$ -skeleton of (D) , and the corresponding Schwartz index of such frames: just as the concept of Schwartz index can be extended to stratified vector fields in general (section 1), so too one can define the Schwartz index of stratified frames in general, and use any such frame to define the corresponding Schwartz class (see [6, 11]).

The second extension of the concept of tangent bundle is given by the Nash bundle $\tilde{T} \rightarrow \tilde{V}$ over the Nash Transform \tilde{V} , which is biholomorphic to V_{reg} away from the divisor $\nu^{-1}(Sing(V))$, where $\nu : \tilde{V} \rightarrow V$ is the projection. Thus \tilde{T} can be regarded as a bundle that extends $T(V_{reg})$ to a bundle over \tilde{V} . The Chern classes of \tilde{T} lie in $H^*(\tilde{V})$, which is mapped into $H_*(\tilde{V})$ by the Alexander homomorphism (see [4, 5]); the Mather classes of V , introduced in [46], are by definition the image of these classes under the morphism $\nu : H_*(\tilde{V}) \rightarrow H_*(V)$. MacPherson's Chern classes for singular varieties [46], lie in the homology of V and can be thought of as being the Mather classes of V weighted (in a sense that is made precise below) by the local Euler obstruction (§2 above). In fact, it is easy to show that the local Euler obstruction satisfies that there exists unique integers $\{n_i\}$ for which the equation

$$\sum n_i Eu_{\nabla_\alpha}(x) = 1 \quad (6.1)$$

is satisfied for all points x in V , where the sum runs over all strata V_α containing x in their closure. Then the *MacPherson class* of degree r is defined by:

$$c_r(V) = c_r^M(\sum n_i \overline{V}_\alpha) = \sum n_i \iota_* c_r^M(\overline{V}_\alpha),$$

where $c_r^M(\overline{V}_\alpha)$ is the Mather class of degree r of the analytic variety \overline{V}_α .

MacPherson's classes satisfy important axioms and functoriality properties conjectured by Deligne and Grothendieck in the early 1970's.

Later, Brasselet and Schwartz [8] proved that the Alexander isomorphism $H^*(M, M \setminus V) \cong H_*(V)$, carries the Schwartz classes into MacPherson's classes, so they are now called the *Schwartz-MacPherson classes* of V . As we briefly explained before, Schwartz classes are defined via the Schwartz indices of vector fields and frames; the MacPherson classes are defined from the Chern classes of the Nash bundle (which determine the Mather classes) and the local Euler obstructions.

A third way of extending the concept of tangent bundle to singular varieties was introduced by Fulton and Johnson [24]. The starting point is that if a variety $V \subset M$ is defined by a regular section s of a holomorphic bundle E over M , then one has the virtual tangent bundle $\tau V = [TM|_V - E|_V]$, introduced in §4 above. The Chern classes of the virtual tangent bundle τV (cap product the fundamental cycle $[V]$) are the *Fulton-Johnson classes* of V . One has ([42, 56]) that the 0-degree Fulton-Johnson class of such varieties equals the total virtual index of every continuous vector field with isolated singularities on V_{reg} . Similar considerations hold for the higher degree Fulton-Johnson classes, using frames and the corresponding virtual classes (see [6, 11]).

Summarizing, the various indices we presented in sections 1-4 are closely related to various known characteristic classes of singular varieties that generalize the concept of Chern classes of complex manifolds. There is left the homological index of section 5: this ought to be related with a fourth way for extending the concept of tangent bundle to singular varieties (with its corresponding generalisation of Chern classes), introduced and studied by Suwa in [58]. This is by considering the tangent sheaf Θ_V , which is by definition the dual of Ω_V , the sheaf of Khäler differentials on V introduced in §5. The latter is defined by the exact sequence:

$$I_V/I_V^2 \xrightarrow{f \mapsto df \otimes 1} \Omega_M \otimes \mathcal{O}_V \rightarrow \Omega_V \rightarrow 0,$$

and $\Theta_V := \text{Hom}(\Omega_V, \mathbb{C})$. Both sheaves Ω_V and Θ_V are coherent sheaves and one can use them to define characteristic classes of V that coincide with the usual Chern classes when V is non-singular. In particular, if V is a local complete intersection in M , then one has a canonical locally free resolution of Ω_V and the corresponding Chern classes essentially coincide with the Fulton-Johnson classes, though the corresponding classes for Θ_V differ from these. Recent work of J. Schürmann points out in this direction, at least if one considers the homological index of 1-forms.

We refer to [5] for a rather complete presentation of characteristic classes of singular varieties, including the constructions of Schwartz and MacPherson that we sketched above, and to [11] for a discussion of indices of vector fields and their relation with characteristic classes of singular varieties, much deeper than the one we presented here.

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